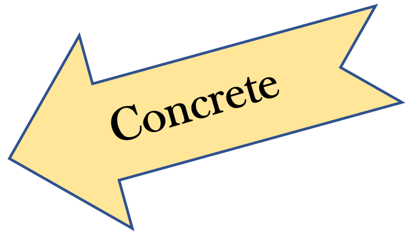
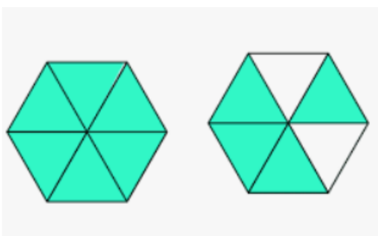
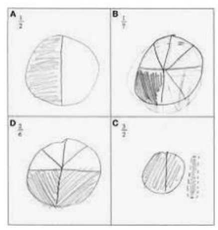
Concrete: Fraction Circles, Fraction Strips



Representation (also called pictorial): Pictures, Models, Drawings

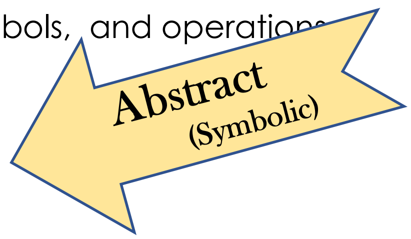




Abstract (also called symbolic) : Using numbers, symbols, and operations

$$\frac{5}{3} + \frac{2}{3} = \frac{7}{3}$$

$$\frac{1}{3} + \frac{1}{2} = \frac{\overset{2+3}{5}}{6}$$



In a balanced math model, students work with manipulatives to directly observe relationships and results. This is part of "conceptual understanding". Later, they move to representations, such as diagrams, drawings, and digital images. Finally, we connect the concrete models and representations to the abstract mathematics; numbers, operations, algorithms. Once a student understands the concept, we can work for fluency with facts and algorithms.

Images: http://mathsmaterials.com/fraction-manipulatives/
https://www.mathportal.org/math-tests/pre-algebra-tests/fractions.php?testNo=1
https://www.researchgate.net/figure/Illustration-of-the-most-common-answer-when-pupils-were-asked-to-draw-a-representation-of_fig2_257890114

There is a misconception by parents and teachers that the models and drawings *simplify* the math, which is sometimes true, and that students that are already fluent do not need the models and manipulatives. This is untrue for two reasons:

1. It is easy to teach procedures and rules, and as long as students can memorize, mimic, and follow steps, they may seem successful, but they often lack the understanding of what is actually happening. This is surface learning, and it is short term (students forget, because these abstract symbols and steps have no meaning). They will not apply their math knowledge and will not develop flexible reasoning.

2. In some situations, the symbolic approach is easier than models: For example, it easy to multiply fractions, but to draw an area model is much more difficult. It is easy to follow steps for fraction division, but to actually *model* fraction division is difficult. It is easy to learn how to multiply binomials, using steps and "tricks" like "FOIL", but to actually understand this multiplication and model it as an area model takes deeper thinking.

In short, modeling **clarifies** mathematical operations. Multiplication is "groups of", which become arrays, which become area models, that support binomial multiplication in algebra. Some students only learn multiplication as repeated addition and then memorizing "facts", which leaves them short changed as they move up through the grades to more complex mathematics.

We see evidence of purely procedural learning in classrooms all the time. Students that apply algorithms but don't understand what they're doing, "carrying" when there is no ten to regroup, failing to align decimals or aligning when they don't need to, adding denominators, missing terms in binomial squares, etc.

Furthermore, many of our outcomes say students will demonstrate "concretely, pictorially, and symbolically", in order to ensure students develop conceptual understanding, but often students are not given the opportunity to model with concrete materials, especially at older grades, or challenged to model fraction operations with diagrams and number lines, as our curriculum indicates.

An overreliance on worksheets can contribute to students memorizing procedures. Classrooms must be structured in a way that students can use manipulatives, have conversations, and explain reasoning.