Factors and Multiples Across the Grades

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Les facteurs et multiples sur toutes les années d'études

A resource for strengthening core numeracy concepts

CTTCS 2023

Rationale:

Many algebra skills in grades 9-12 are more difficult for students who do not have fluency in recognizing factors and multiples

Why factors and multiples?

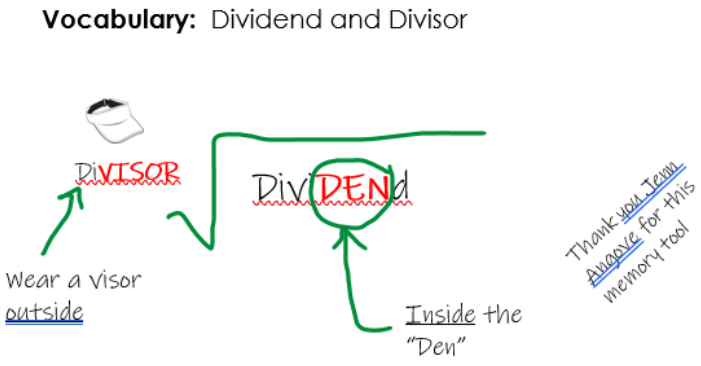
-Ability to multiply and divide
-recognizing divisibility properties
-recognizing multiples so students can simplify fractions
-create common denominators to add fractions
-reduce fractions while multiplying (increase efficiency)
-Find areas, side lengths
-work with ratios, rates, and scales
-factor polynomials
-graph quadratics
-Simplify equations, manipulate equations to find points of intersection
-graph polynomial functions
-Find x intercepts
-simplify radicals
-work with roots and powers
-work with trig functions, trig identities, and proofs
-calculus

This is a core concept of basic numeracy. Understanding numbers means understanding the building blocks of numbers. Numbers can be composed and decomposed into addends, or into factors. Many algebra skills in grades 9-12 are more difficult for students who do not have fluency in recognizing factors and multiples. Students that lack fluency in multiplication work more slowly, and become frustrated more easily as the brain is tied up with computation and students cannot progress to think more deeply about the mathematics and more complex relationships.

Many students in middle years find computations with fractions difficult. Further, we see students able to perform operations, then when we introduce fractions they often will not attempt the problem. Similarly, when introducing algebra, students often shut down if fractions appear in equations. Fluency in factoring and recognizing multiples will enhance fluency in operation on fractions and many other areas, freeing up cognitive processing to think about relationships, graphs, and contexts.

CTTCS screeners have indicated significant gaps in student grade-level understanding of factors and multiples, and as a result, division and divisibility patterns.

What we notice in students:
Lack of fluency in multiplication, and even greater lack in division.
Not recognizing multiplication and division as inverse operations.
Lack of continuity of vocabulary grade-to-grade.
Ability to multiply if the problem is given explicitly, yet unable to divide (shut down).
No connection between fraction form and an expression of division.
Avoidance of fraction problems.
Self declared: "I can't divide". "I can't 'do' fractions".
Inability to recognize multiples—required to simplify ('reduce') fractions, and remove GCF from algebraic expressions.
Inability to recognize GCF to simplify ('reduce') fractions, and understand ratios and scales.
Students do not understand prime vs composite numbers, or the terminology.
Students may not have been introduced to multiplication as repeated addition, and repeated addition connected to arrays. They may not have seen division as related to repeated subtraction.
Students may not have been exposed to arrays as a way of displaying factors, or illustrating prime vs composite numbers, and linking arrays to area.
Inability to fluently identify LCM to make a common denominator.
Inability to manipulate equations to eliminate fractions and decimals.
Students in upper middle years and even high school relying on calculators to multiply basic facts, including multiplication by 10's.
Students in pre-algebra and algebra unable to manipulate equations and graph functions, or too tied up in computation to think about the meaning of the graph and how it is a representation of a relationship.



Curriculum:

Grade 3: Multiplication to 5X5, Division.

3N.7

la notion de multiplication, jusqu'à 5 × 5, y compris : groupes égaux; de matrices (arrangements rectangulaires); créer et résoudre des problèmes connexes; modéliser l'action de multiplier de façon concrète et imagée, et en noter symboliquement le processus ; établir le lien entre la multiplication et la division.

3N.8

la notion de division (se limitant aux faits de division correspondants jusqu'à 5 × 5), y compris : partages en parties égales; regroupements égaux ou de mesures égales; créer et résoudre des problèmes contextualisés connexes; modéliser l'action de diviser de façon concrète et imagée, et en noter symboliquement le processus ; établir un lien entre la division et la multiplication.

Grade 4: Multiplication and Division

Arrays

Vocabulary: Array, multiple, factor, product, related facts (fact families)

4N.5

Décrire et appliquer, pour déterminer les faits de multiplication jusqu'à 9 × 9 et les faits de division reliés, des propriétés des nombres et des stratégies de calcul mental, telles que :

- la notion de doubler et d'ajouter ou d'enlever un ou deux groupes;
- la notion de doubler et de diviser par 2;
- les régularités qui se dégagent des faits de multiplication par 9;
- les doubles répétés;
- les carrés;
- le compte par sauts à partir d'un fait connu;
- le lien entre la division et la multiplication;
- la propriété de la commutativité;
- les propriétés de 0 et de 1 pour la multiplication et la division;
- la division d'un nombre par le même nombre (sauf 0).

Grade 5: Recall multiplication facts to 9X9

Multiplication by 1 and 0

Multiplication by factors of 10

Arrays, but also area model

Strategies to multiply, including skip counting, halving, repeated doubling, decomposing and using distributive property/partial products ex: 5 X 14 = 5(10 + 4) which is (5 x10) + (5 x 4)

2 digit multiplication strategies (partial products, mental math as well as algorithms)

Use of base ten blocks to model concretely. Use C-R-A continuum to link abstract representation to concrete models.

Long division-3 digit by one digit

Recall division facts to dividend of 81

Understand and explain division by 0

Remainder and choices of how to report it

Clarification of zero remainder indicating the divisor is a "factor" of the dividend

Vocabulary: Array, distributive property, partial product, decompose, factors, multiples, partial product, expanded notation, area, divisor, dividend, quotient, remainder

5N.3

Développer et appliquer des stratégies de calcul mental et des propriétés du nombre pour déterminer avec fluidité les faits de multiplication jusqu'à 81 et les faits de division correspondants, telles que :

- le compte par sauts à partir d'un fait connu;
- la notion de doubler ou de diviser par deux;
- les régularités qui se dégagent des faits de multiplication ou de division par 9;
- les doubles répétés ou les moitiés répétées;
- les carrés;
- la propriété de la commutativité;
- les propriétés de zéro et de un.

5N.4

Approfondir et appliquer, avec ou sans l'aide de matériel concret, sa compréhension de la notion de multiplication pour multiplier des numéraux à deux chiffres par un numéral à deux chiffres, y compris :

- des stratégies de calcul mental;
- ses propres stratégies d'estimation et de calcul;
- des propriétés du nombre;
- la résolution de problèmes contextualisés connexes.

5N.5

Approfondir et appliquer, avec ou sans l'aide de matériel concret, sa compréhension de la notion de division pour diviser des numéraux (dividende de numéraux à trois chiffres par un diviseur à un chiffre), y compris :

- le partage et le regroupement égal;
- ses propres stratégies;
- des stratégies de calcul mental et des propriétés du nombre;
- ses stratégies d'estimation;
- l'interprétation des restes;
- la résolution de problèmes contextualisés connexes.

5N.6

Démontrer à l'aide de représentations concrètes et imagées une compréhension de la notion de fraction équivalente, y compris :

- créer des ensembles de fractions équivalentes;
- comparer et ordonner;
- établir le lien entre la représentation concrète et imagée et la représentation symbolique;
- résoudre des problèmes contextualisés connexes.

Grade 6 The first year *factors and multiples are explicitly taught*, though students should be familiar with these words from grades 3,4,and 5

Find factors and multiples of numbers less than 100
Relate skip counting to multiples
Prime and composite numbers
Factors and Prime factors
Using prime factors to determine composite factors
Find common factors, greatest common factor
Find multiples and least common multiple
Multiplying and dividing decimals
Mixed and improper fractions (relates to remainders, fractions as division, possibly GCF if students experience equivalent fractions)

Vocabulary: factor, multiple, divisible, prime, composite, factor (as a verb), prime factorization, arrays, product, dividend, divisor, quotient, skip counting, GCF, LCM

6N.2
Démontrer de façon concrète, imagée ou symbolique une compréhension de la notion de facteur et de multiple, y compris :

- déterminer des multiples et des facteurs de nombres inférieurs à 100;
- établir le lien entre les facteurs et les multiples de nombres inférieurs à 100;
- déterminer des nombres premiers et des nombres composés;
- établir le lien entre les nombres premiers et les nombres composés;
- résoudre des problèmes contextualisés connexes.

6N.3
Appliquer de façon concrète, imagée et symbolique sa compréhension de la notion de fraction aux fractions supérieures à un et aux nombres fractionnaires. (C, CE, L, R, V)

6N.5
Appliquer sa compréhension de la notion de multiplication et la notion de division aux nombres décimaux où le multiplicateur est un nombre entier positif à un chiffre (0 à 9) et le diviseur est un nombre entier strictement positif à un chiffre (1 à 9). (C, CE, L, R, RP, V)

Grade 7 Divisibility rules are taught
Determine or validate factors using divisibility rules
Multiply and divide decimals
Write equivalent fractions and reduce (simplify) fractions—requires application of factors and GCF. *Reducing fractions requires recognizing multiples.*
Add and subtract fractions (requires application of LCM -Least common multiple-- as LCD lowest/least common denominator)

Fractions, decimals, and percents—involves equivalent fractions and ratios with denominator of 100, factors of 100.

Vocabulary: factor, multiple, divisible, prime, composite, factor (as a verb), prime factorization, arrays, product, dividend, divisor, quotient, skip counting, GCF, LCM. LCD, reducing (simplifying), fraction, ratio, percent, equivalent

7N.1
Appliquer sa compréhension de la notion de division en vue de (d') :

- développer et utiliser des stratégies pour déterminer et préciser la divisibilité par 2, 3, 4, 5, 6, 8, 9 ou 10;
- analyser la division de zéro par un nombre;
- expliquer pourquoi on ne peut pas diviser un nombre par zéro.

7N.2
Appliquer sa compréhension de la notion d'addition, de soustraction, de multiplication et de division aux nombres décimaux et pour suivre l'ordre des opérations sans puissances. (C, CE, L, R, RP, T)

7N.3
Démontrer une compréhension du lien entre les nombres naturels, les nombres décimaux positifs, les fractions positives (y compris les nombres fractionnaires, et les fractions supérieures à un). (C, CE, L, R, T)

7N.4
Appliquer de façon concrète, imagée et symbolique sa compréhension de la notion d'addition et de soustraction aux fractions positives et aux nombres fractionnaires positifs, avec ou sans dénominateurs communs, se limitant aux sommes et aux différences positives. (C, CE, L, R, RP, V)

7N.5
Démontrer une compréhension de la notion de pourcentage de 1 % à 100 % (se limiter aux nombres naturels). (C, L, R, RP, V)

Grade 8 Using arrays and area models to model factors, composite and prime numbers
Factor numbers
Determine principle square root of numbers, identify square numbers
Multiply and divide fractions—involves canceling common factors from numerators and denominators (reducing/simplifying)
Exponent rules
Relating fractions, decimals, percents, ratios, rates and proportions
Multiplication and division of integers

Vocabulary: factor, multiple, divisible, prime, composite, factor (as a verb), prime factorization, arrays, product, dividend, divisor, quotient, skip counting, GCF, LCM. LCD, reducing (simplifying), fraction, ratio, percent, equivalent, prime, composite, area, coefficient, variable, exponent

8N.1
Démontrer de façon concrète, imagée et symbolique une compréhension de la notion de racine carrée (se limitant aux nombres naturels).

8N.2
Approfondir et appliquer de façon concrète, imagée et symbolique sa compréhension de la notion de pourcentage aux pourcentages fractionnaires et décimaux supérieurs ou égaux à 0 %, y compris supérieurs à 100 %.

8N.3
Démontrer de façon concrète, imagée, symbolique et à l'aide de raisonnement proportionnel une compréhension des notions de rapport et de taux.

8N.4
Approfondir et appliquer de façon concrète, imagée et symbolique sa compréhension de la notion de multiplication et de division aux fractions positives et aux nombres fractionnaires positifs.

8N.5
Approfondir et appliquer de façon concrète, imagée et symbolique sa compréhension de la notion de multiplication et de division aux nombres entiers (positifs et négatifs).

Grade 9 Exponents, powers of positive and negative bases and rational bases (requires multiplication fluency). Square roots of rational radicands.
Add, subtract, multiply and divide fractions (requires recognizing multiples, common factors, common multiples) and order fractions, decimals and percents.
Solving equations, which includes eliminating fractions in an equation (using LCM of the denominators) and removing GCFs from equations (recognizing multiples).
Multiplying and dividing polynomials
Scale factor, similar figures

Ex: To solve $\frac{x}{2}-\frac{3}{5}=-\frac{2}{3}x+\frac{1}{6}$ students need to find the LCM of 2,5,3, and 6 and multiply each of the four terms by that number.

Vocabulary: Exponent, base, square root, radicand, equivalence, square (perfect square), LCD, reduce (simplify), factors, multiples, term, coefficient, like terms, expand, distribute, binomial multiplication, similar figures.

Grade 10
Application of all skills listed –multiplication, division, fluency with fractions, but extended to:
Factoring polynomials –sum and product factoring requires being able to identify factors of a product, both positive and negative possibilities, that sum to a given quantity. This skill is applied throughout algebra, graphing, precalculus, and calculus.

Ex: To factor $x^2 - x - 20$ a student would look for factors of -20 that sum to -1. These factors allow the student to determine the roots or x intercepts of the function.

Powers and roots, including cube, fourth, fifth and 6th roots (requires understanding of factors and prime factors), simplifying radicals (recognizing multiples)

Ex: to simplify $\sqrt[3]{54}$ a student would need to be familiar with perfect cubes and recognize 54 as a multiple of 27.

Introducing vocabulary early

Research has shown the importance of explicitly teaching vocabulary in math. We need to say terms, encourage proper terminology in mathematical dialog, and ensure that math teaching teams through grades communicate to establish consistency and flow of vocabulary from grade to grade.

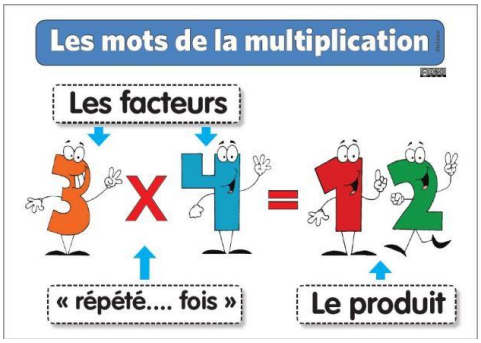
In Grades 3, 4, and 5 as we introduce students to arrays, we use concrete representations and visual models that are then translated to representations:

3N.7
- une équation (phrase numérique)
- des groupes égaux ou des matrices (représentations rectangulaires)
- le lien entre la multiplication et la division

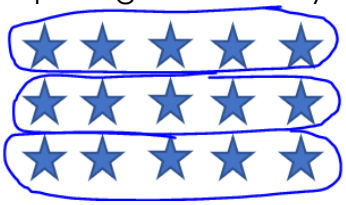
3N.8
- partages en parties égales
- regroupements égaux ou de mesures égales
- le lien entre la multiplication et la division

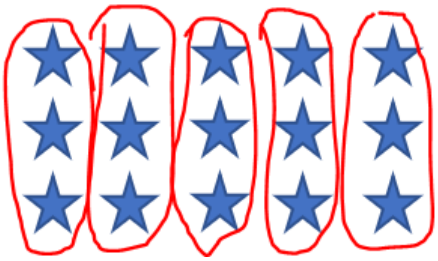
We write $\mathbf{3 \times 5 = 15}$ and $\mathbf{5 \times 3 = 15}$. **As we introduce multiplication it is important to immediately introduce the division facts:**
$\mathbf{15 \div 5 = 3}$ and
$\mathbf{15 \div 3 = 5}$.
In our curriculum, these are called "related facts". Our data shows that we have students that are comfortable with multiplication facts but do not attempt division questions. We can help with this by always connecting multiplication to division.

4N.5
- les faits de multiplication jusqu'à 9 × 9 et les faits de division reliés
- des propriétés des nombres et des stratégies de calcul mental



Two numbers that multiply each other are called "factors". The result is called a "product". Students are used to decomposing numbers by addition: $10 = 2 + 8$ or $6 + 4$, etc. Now they are seeing that we can decompose numbers by factors.

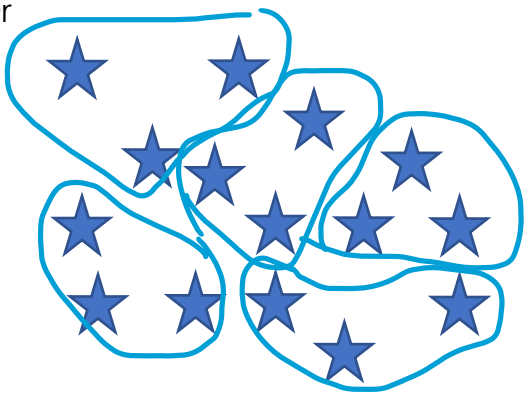




Help students see that multiplication is "groups of"

15 is 5 "groups of" 3 or "groups of 5" (3 and 5 are *factors* of 15. 15 is a *multiple* of 3, and a *multiple* of 5. 15 is *divisible* by 3 and by 5, etc)

Or

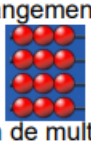


15 is 5 groups of 3 (random arrangement). $3 + 3 + 3 + 3 + 3 = 5$. Multiplication is a shorter way of writing repeated addition (mathematical connection).

3N.7
(g) Établit le lien entre la multiplication, le compte par sauts de 1, 2, 3, 4 et 5 et les groupes égaux.
3N.8
(g) Établit le lien entre la division et le compte par sauts à rebours.

Le lexique et les symboles de mathématiques

Une disposition rectangulaire – un arrangement de rangées et de colonnes d'un ensemble d'objets, de symboles, ou de nombres.



Un produit – le résultat d'une opération de multiplication (p. ex., 6 est le produit de 2 x 3).
X est le symbole de multiplication.
3 x 2 a de nombreuses interprétations, notamment :

- Trois fois deux
- Trois, deux fois
- Trois groupes de deux

Ask: Does it matter in what order we multiply our factors? (No. This is called the "commutative property" of multiplication.
Is division commutative? Is addition? Subtraction?)
To "commute" to work means I live in one place and work in another. I trade places. The commutative property of addition and multiplication mean the numbers can trade places and we do not change the value. Helping students see mathematical terminology connected to other things they understand and other subject areas is a

4N.5

- le compte par sauts à partir d'un fait connu;
- le lien entre la division et la multiplication;
- la propriété de la commutativité;

mathematical connection (one of our mathematical processes) and is called teaching "generative vocabulary"—showing where words come from. This is the most effective form of vocabulary instruction.

> "Mathematics presents challenging reading because this content area has **more concepts per word, per sentence, and per paragraph** than any other area" (Harmon, Hedrick, & Wood, 2005).

A note about Division: Did you know there are two kinds of division? No wonder our kids get confused!

Partitive division: Means "equal sharing". In this type of division we state the number of groups first, and determine how many elements are in each group:
Ex: share 12 cookies equally onto four plates:

$12 \div 4 = 3$

3 means the number of cookies in each group/plate

Number of groups

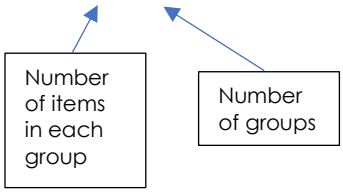
Number of items in each group

Exemple de la division partition, ou division-partage : J'ai 24 œufs en chocolat, et je veux en donner de façon équitable à 6 enfants. Combien chaque enfant recevra-t-il d'œufs ?

But wait!
Quotative Division: I have 24 horseshoes left. How many horses can I shoe? Each horse needs four shoes:

So now I'm looking for how many groups of 4 are in 24? I've already designated how many elements are in each group; my answer this time is how *many groups*.

$24 \div 4 = 6$



Exemple de la division quotition, ou division-groupement : J'ai 24 œufs en chocolat, et je prépare des sachets de 4 œufs. Combien vais-je remplir de sachets ?

Mathematically, there is no difference, but contextually it's a huge difference! We need to clarify this for students. Sometimes in division I say how many groups we need, you tell me how many items in each group (partitive). But sometimes I say how many items are in a group, and you tell me how many groups of that size I can make out of a given number (quotative).

In grade 5, students are making arrays, studying strategies for multiplication (doubling, doubling twice to multiply by four, doubling three times to multiply by 8).

If I double something three times that's X 2, X 2, and X 2 again
$2 \times 2 \times 2 = 8.$
Students learn to multiply by ten. Multiplying by 5 is multiplying by ten and taking half.

Développer et appliquer des stratégies de calcul mental
5N.3
- pour déterminer avec fluidité les faits de multiplication jusqu'à 81 et les faits de division correspondants

Approfondir et appliquer, avec ou sans l'aide de matériel concret
5N.4
- sa compréhension de la notion de multiplication pour multiplier des numéraux à deux chiffres par un numéral à deux chiffres

5N.5
- sa compréhension de la notion de division pour diviser des numéraux (dividende de numéraux à trois chiffres par un diviseur à un chiffre)

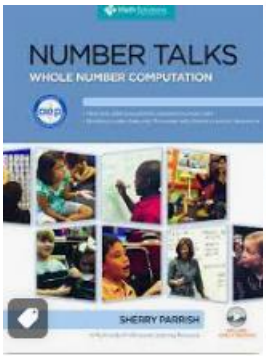
<u>Building fluency</u>
We can list multiplication facts by listing multiples which is skip counting (also connects to repeated addition). Skip counting builds fluency in multiplication.

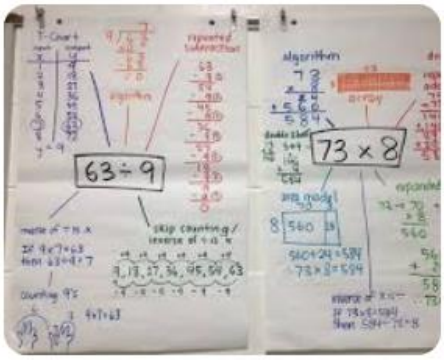
We can work directly on fact fluency by practicing a little bit frequently, and using *spaced practice*. Do not time practice.

There are lots of games that build multiplication fluency! It's easy to find games on line, but face to face games are important, as there is a social aspect to learning, kids share strategies, they can use concrete manipulatives, and most kids have enough screen time before they get to us.

Multiplication fact fluency is something easy that parents can help support at home. Don't forget to continually teach the corresponding division facts when doing multiplication.

One of the best ways to build fluency and flexible reasoning is through Number Talks. If you're not familiar with these, contact your ILC!

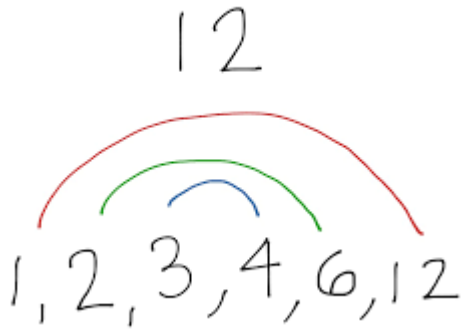




Vidéo : **Formation sur les bavardages mathématiques** (YouTube ; CSSH – Services éducatifs, 2020)

Introducing Factoring –Grade 6

Rainbow method:
https://youtu.be/vqYhXDFuF0M



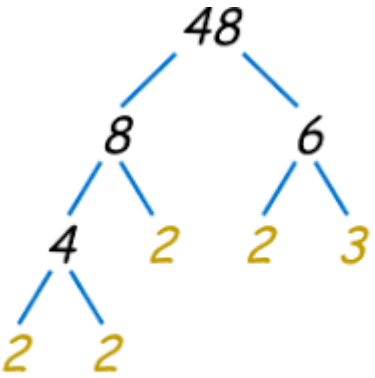
Vidéo : **Les arcs-en-ciel de facteurs** (YouTube ; NLESD Mathématiques, 2018)

The advantages of this method are:
-shows ALL the factors, in order
-highlights square numbers (they'll have 2 identical factors in the middle)

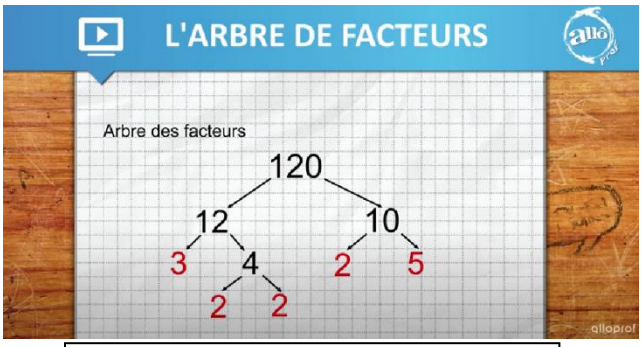
This method does not show *prime factors*.

Prime factorization:
This is an important skill and will be useful in high school algebra.

Tree method:
https://youtu.be/tW97UU01ShY

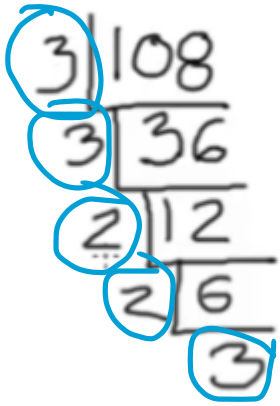


48 = 2 x 2 x 2 x 2 x 3



Vidéo : **Allô prof - L'arbre de facteurs** (YouTube ; Alloprof, 2015)

An older prime factorization method (not shown in MMS books)
https://youtu.be/5HCrfzSC0FM



Once we know the prime factors, or building blocks of a number, we can group them to build all other factors! Also, perfect squares, cubes, fourth powers etc are revealed.

$108 = 3 \times 3 \times 2 \times 2 \times 3$
(or $3^3 \times 2^2$ in Gr 9)

Some students like this better. It's more orderly, and arguably quicker.

A note about vocabulary:
Note that we already need to know "prime" from "composite" numbers.

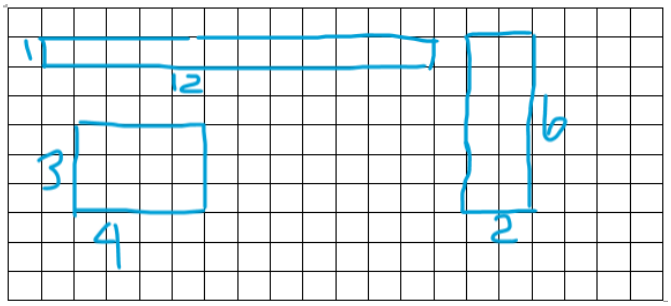
What is the difference between prime and composite?

A prime number is a number which has exactly two factors i.e. '1' and the number itself. A composite number has more than two factors, which means apart from getting divided by 1 and the number itself, it can also be divided by at least one positive integer.

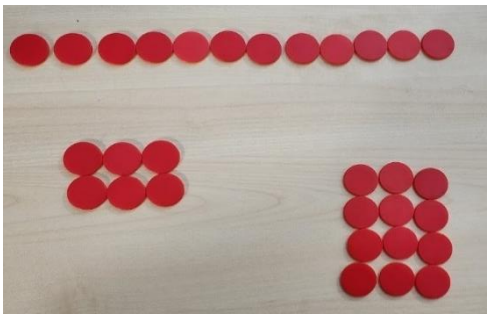
Vidéo : **Qu'est ce qu'un nombre premier ?** (YouTube ; Lumini, 2022)

A great way to introduce prime numbers, composite numbers, and factors is using arrays.

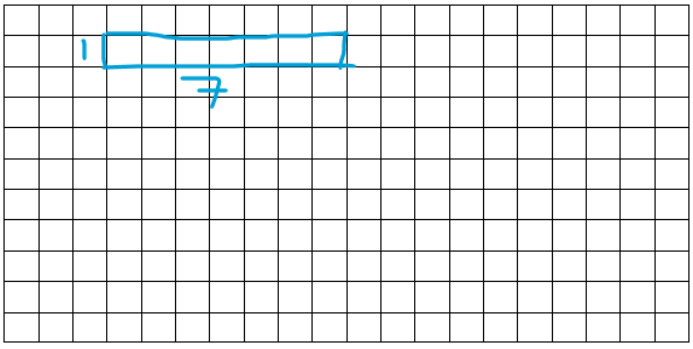
Give students square tiles or counters, or ask them to draw lines on graph paper. How many rectangles can you make with area of 12 square units? Note that this shows all the factors of 12.

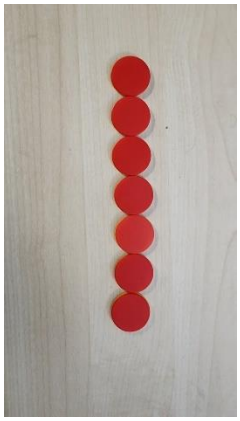


Or



Now how many rectangles can we make with area = 7?





Only one—a 1 X 7 array.
7 is prime. There is only one array for 7.

Even more confusing vocabulary with division:

Ever hear of the "goes-intos"? https://youtu.be/1H8e0MMwUec

Sometimes we say to students "well five "goes into" twenty" or we ask "how many times does five "go into" twenty but we never explain what we mean.

More correctly, we say "twenty is divisible by five" or "how many groups of five are in twenty".

If we use "goes into", **which is slang**, let's explain what we mean. They're going to hear division spoken about this way, but no text would ever use this language.

Furthermore, once we get to long division, the language of the algorithm confuses students, because we tend to read left to right (though in math it is sometimes necessary to read both ways, which makes the teaching of inequality signs troublesome).

So if we say "352 divided by 4" some students will set up the algorithm like this

because they are recording what we say, left to right.

Similarly, some people read the algorithm  as "four *into* 352"

without explaining that slang terminology to kids—that "four into 352" means *352 divided by four.*

Ask these questions:
"Is 4 a **factor** of 352?" "How do you know?"
"Is 352 **divisible** by four?" "Is 352 a **multiple** of 4?"

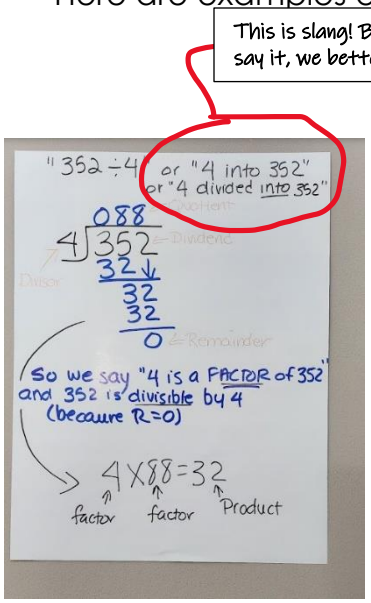
Be sure to ask the questions often, also when the remainder isn't zero.

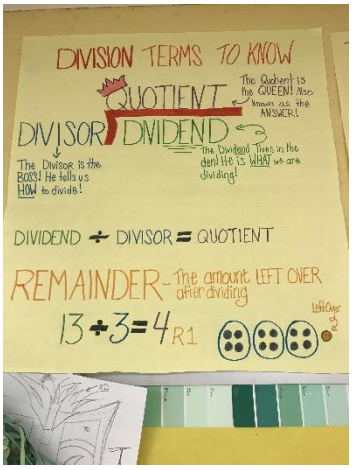
Ask for the related multiplication fact.

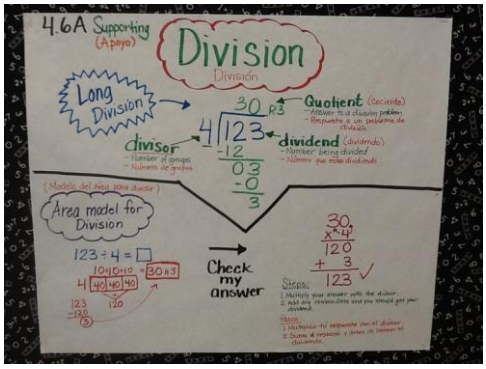
Teach the vocabulary *divisor, dividend, quotient, remainder* by labeling the algorithm.

Use these words as often as possible, across grades, to ensure students understand the meaning of the words. Make anchor charts with visual definitions of the words, labeled algorithms, etc.

Here are examples of anchor charts/posters





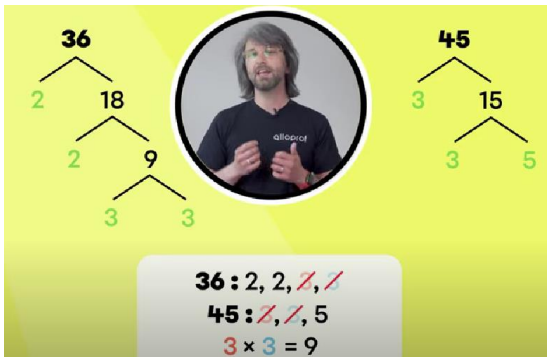


Why are prime factors useful?

We can find prime factors of anything, and use them to find the least common multiple (LCM) and greatest common factor (GCF)

https://watch.screencastify.com/v/oVTndNLaTtHWntWOGexa

Vidéo : **Le PPCM : les méthodes des multiples et de l'arbre des facteurs** (YouTube ; Alloprof, 2018)

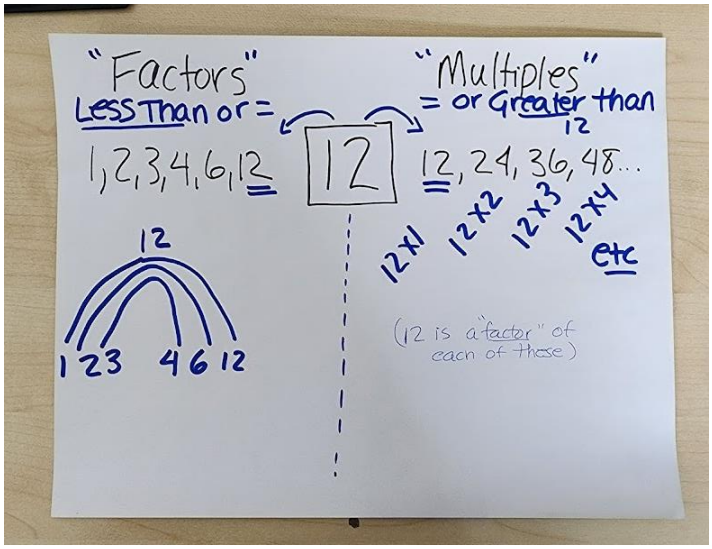


Vidéo : **Le plus grand commun diviseur (PGCD)** (YouTube ; Alloprof, 2021)

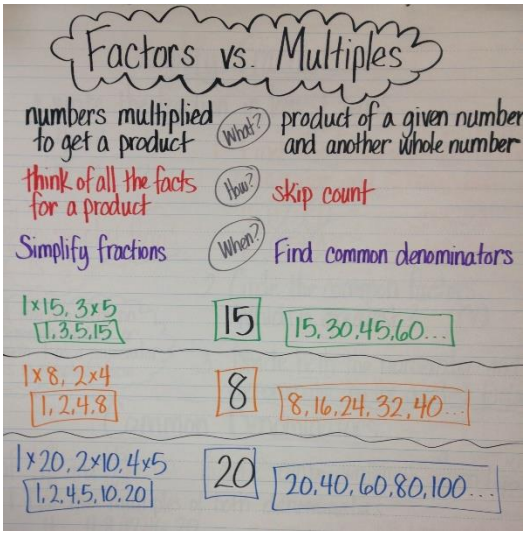
Prime factors are also useful for determining roots
https://watch.screencastify.com/v/ntaZ9dsvy0mEv7ujxksn

Another handy anchor chart

Factors are the facts. They are FEW



Multiples are many (infinite number!)



Recognizing Multiples

Even when students are fluent with multiplication and division facts, can find factors and determine common multiples, a weakness lies in recognizing common multiples. In other words, presented with a fraction like $\frac{36}{60}$, many of our students are unable to *recognize* the numerator and denominator as multiples of 12.

When it comes to fractions, students get by, because they may reduce this piece by piece:

$$\frac{36}{60} \div 6 = \frac{6}{10} \div 2 = \frac{3}{5}$$

So they may recognize the common factor of 6, and then 2, or some students may do 2, then 2 again, then 3. We need to make sure to point out to students that if the numbers were divisible by 6 then 2, or 2,2,then 3, they must have been divisible by 12. So students find common factors, but not greatest common factors, which is less efficient. Some students struggle even to recognize both numbers as even, and therefore divisible by 2—or they stop before they remove all the common factors.

Recognizing multiples is required in other areas:
In grade 10 a student may struggle to graph or find intercepts of a linear function like $4x + 8y + 40 = 0$ when it is expected that they would divide out a common factor of 4. A student may try to factor a trinomial like $6x^2 - 30x - 84$, which is very difficult unless they remove the GCF of 6.

Being able to *find* multiples is not the same as being able to *recognize* common multiples.

Similarly, a student may need to recognize the radicand, 24, in $\sqrt[3]{24}$ as a multiple of 8 in order to simplify the term to $2\sqrt[3]{3}$.

Grade 7: Divisibility rules

In this outcome, students learn the rule of divisibility by 2,3,4,5,6,9 and 10. In our curriculum this seems like a skill taught in isolation. We need to connect this learning to the bigger idea of factors and multiples. Most importantly, this outcome is getting at **recognizing multiples**, *which is often where skills crash*.

Not only that, the "Divisibility Rules" outcome is often taught as a "One-hit Wonder"—something kids memorize for one unit of study, then forget, because we don't point out for them enough how often we apply them. We can strengthen student fluency by linking divisibility rules to multiplication and recognizing multiples when working with fractions, then later roots and algebra.

Divisibility Rules = recognizing multiples and finding multiplication strategies!

When we say "numbers divisible by 5" do students know that is the same as saying "multiples of 5"?

Students don't study divisibility rules until Grade 7, but we can give them sneak peaks ahead of time to help them! Usually by Grade 5, students can tell you that numbers that are multiples of ten (or numbers that can be divided by 10/are divisible by ten) end in zero. Numbers divisible by 5 end in 5 or 0. They should know that even numbers end in 2,4,6,8, or 10, but probably have not heard the language that "even numbers" are divisible by 2. This is part of our work to strengthen the vocabulary around factors and multiples up through the grades.

Do students have an understanding of what it means if a number is "divisible" by another number?

Numbers divisible by 3 have digits that sum to 3. This is true of multiples of 9 also.

Ask students: Are numbers divisible by ten also divisible by 5? Always or sometimes? Or never? Are they divisible by 2? Always/sometimes/never? Are they also divisible by 3? Always/sometimes/never.

Understanding that numbers divisible 10 are also divisible by 5 and 2 helps us with multiplication strategies: An easy way to multiply something by 5 is to multiply it by ten then take half. What is 18 x 5? Take 18 X 10 which is 180, and cut that in half: 90.

Are numbers that are multiples of four also multiples of two? Yes, twice! So a strategy for multiplying by 4 is to double the number and double it again.

For example: 16×4
Double 16 which is 32, double it again which is 64. Why? Because 16×4 is the same as $16 \times 2 \times 2$

Are multiples of four always even numbers??

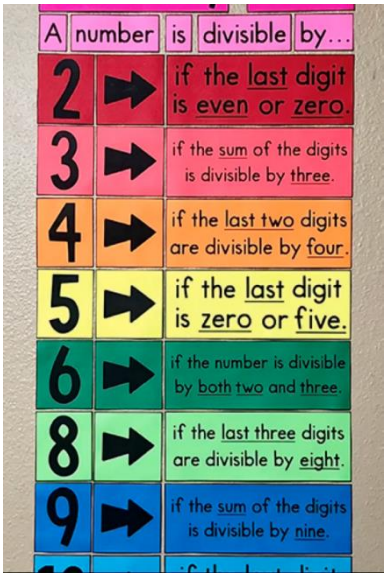
How can I look at a large number and know if it is divisible by 4?
Ex, 65 532 is divisible by 4. Why?

Similarly, eight is $2 \times 2 \times 2$ so multiplying by 8 is double, double, double.

How do I know if something is divisible by 6? Well, 6 is built of 2×3, so if it's an even number (multiple of 2) and divisible by 3 (digits sum to 3) then it is divisible by 6. The number 8 322 is divisible by 6. $8 + 3 + 2 + 2 += 15$ (which is divisible by 3, but if you didn't know you could keep adding digits: $1 + 5 = 6$, divisible by 3) AND 8 322 is even (divisible by 2) so therefore divisible by 6.
Numbers divisible by 9 have digits that sum to 9, but also, its easy to reveal the pattern to the 9 times table, up to 9×9.

https://watch.screencastify.com/v/IsF910miDHDMP4azJS0m

If you're interested, 7 has a pattern also. Take any number: double the last digit, subtract it from the remaining digits: If the answer is a multiple of 7, then the original number is a multiple of 7. For example, 455.
Take the last digit, 5, and double it, which is 10. Subtract that from the other two digits: 45 – 10 is 35, and 35 is divisible by 7, so 455 is divisible by 7. This is not something students really need to learn or memorize, but interesting. It doesn't help us multiply by 7.

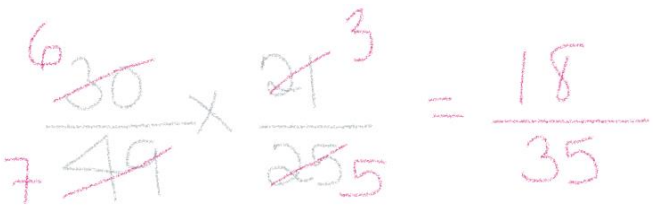


We teach students to generate multiples, by skip counting or multiplication strategies, but there is very little practice with ***recognizing multiples***, which becomes a stumbling block.

See resources in this kit that help students recognize multiples/common factors.

Grade 8: Multiplying fractions:

Of course we want to point students to efficient methods, so canceling out factors before multiplying is the best approach.



If students do not recognize these common factors and eliminate them before multiplying, they are left with

…and then need to try to reduce that. Yuck.

Pour simplifier, un des moyens les plus courants est de chercher les multiples communs des numérateurs et des dénominateurs.

Exemples:

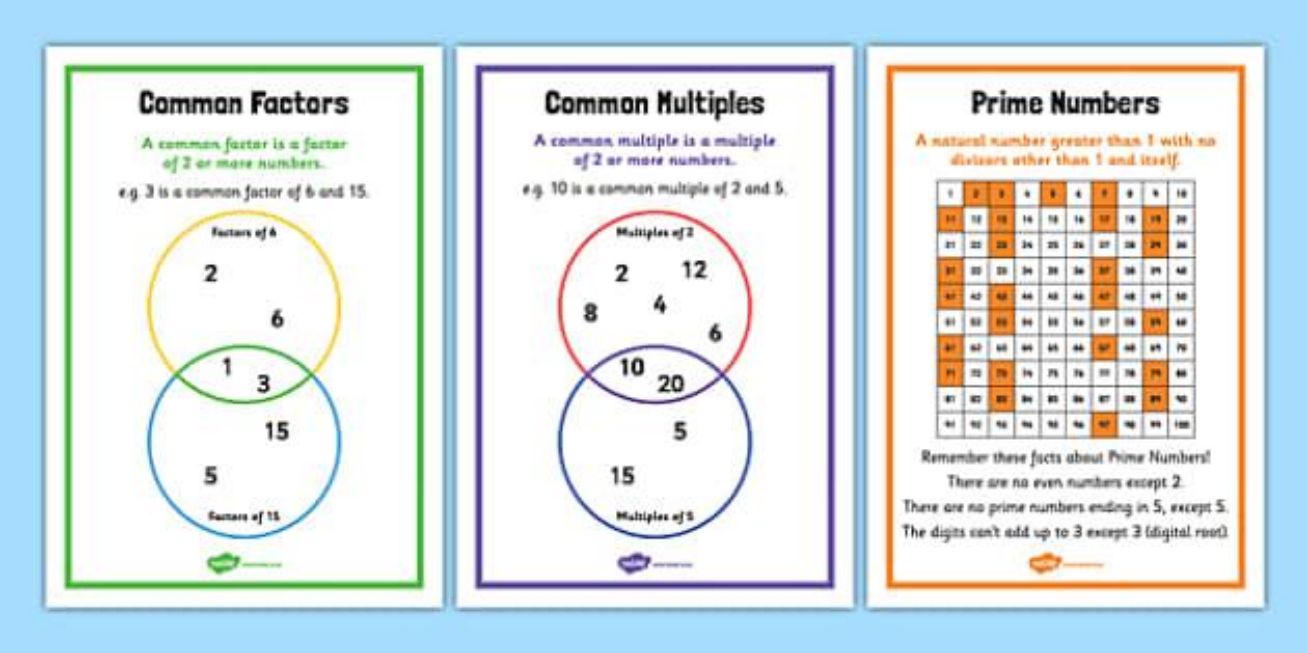
$$\frac{45}{63}\times\frac{8}{20}=\frac{9\times5}{9\times7}\times\frac{4\times2}{4\times5}=\frac{9\times5\times4\times2}{9\times7\times4\times5}=\frac{9\times5\times4\times2}{9\times4\times5\times7}=\frac{2}{7}$$

On peut simplifier par $9\times5\times4$ (ou $9\times4\times5$)

https://www.educastream.com/fr/fractions-multiplication-4eme

(This may require a conversation about what constitutes "one term" and what does not.)

https://watch.screencastify.com/v/Fm2CuqL6lj5VyF4SENZz



Vocabulary

Factor Multiple Prime Composite Divisible

Multiply Divide Common Factor Common Multiple

Greatest Common Factor Least Common Multiple
Lowest Common Denominator

Product Quotient Divisor Dividend Remainder Factorization

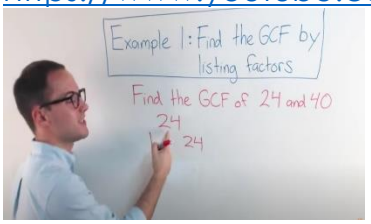
Prime Factors Decompose Algorithm

Grade 4 and 5: Strengthen vocabulary around "factor, multiple, divisible". Use arrays to point out commutative property, and to link multiplication to division. This is accompanied by language: if 4 X 5 = 20 then 4 is a *factor* of 20, and so is 5. 20 is a *multiple* of 4 and a *multiple* of 5. 20 is *divisible* by both 4 and 5,and so on.
Grade 5: Long division requires fluency in division and multiples. Strengthen language around the remainder—if zero remainder, then the divisor is a factor of the dividend, and the dividend is a multiple of both the divisor and the quotient, and the opposite if the remainder is not zero.
Grade 6: Vocabulary around multiples, common multiples, factors, common factors. When "simplifying" fractions (writing equivalent fractions and choosing the one in lowest terms) use vocabulary "common factors in numerator and denominator", "both are divisible by the same number", etc. Factoring and prime factoring outcome can be reintroduced throughout the year. Provide spaced practice.
When changing mixed fractions to improper fractions and vice versa, we can connect to division (fractions as a way of writing division) and long division with remainder an alternative way to creating mixed fractions (remainder over divisor).
Grade 7: Divisibility rules are ways of recognizing common factors, and also support strategies for multiplication. Strengthen mental math and flexible reasoning. Review factoring algorithms and how factors determine divisibility rules. We can strengthen this outcome by making strong connections and applications to working with fractions: reducing fractions (aka eliminating common factors), and finding common denominators (aka least common multiples of denominators).
Grade 8: Efficiently reducing fractions by eliminating common factors before multiplying. Common multiples are part of ratios and rates, similar figures and scale factor.

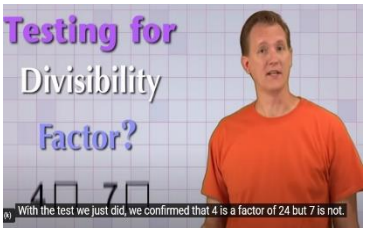
Helpful Videos

3 minutes 45 seconds Shows how to use prime factors to generate LCM and GCF
https://www.youtube.com/watch?v=NtkjbVb3Zv8

2 minutes 20 seconds Using ladder method to find LCM and GCF
https://www.youtube.com/watch?v=3W8SeYgZcMo

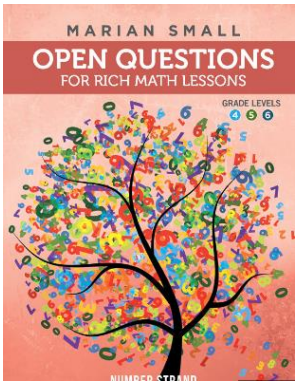


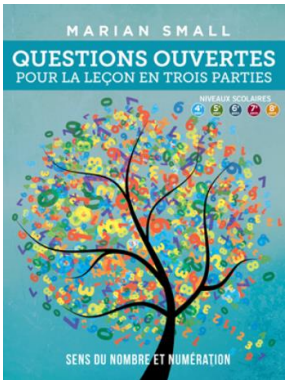
6 minutes 53 seconds Finding factors, LCM and GCF
https://www.youtube.com/watch?v=y4MG3m5uJQg



6 minutes 17 seconds. Divisibility rules, factors and multiples. Math Antics video
https://www.youtube.com/watch?v=0NvLtTwnUHs

Resources:
Book Open Questions Number Strand by Marian Small





A la bibliothèque du conseil scolaire :
QUESTIONS OUVERTES :
GRADES K-3, 4 – 6, 7-9

- DOMAINE DU NOMBRE
- LA FORME ET L'ESPACE
- LES RÉGULARITÉS ET LES RELATIONS

Factors and multiples game https://slideplayer.com/slide/12547402/
Factors and Multiples

Juniper Green http://manghammath.com/Activities/Juniper%20Green.pdf
How to play:
https://themathbehindthemagic.wordpress.com/2021/02/26/juniper-green/#:~:text=To%20play%20Juniper%20Green%2C%20you,of%20the%20previous%20player's%20choice.

Online practice
Recognizing multiples!!!! https://www.topmarks.co.uk/times-tables/coconut-multiples

GCF
https://mrnussbaum.com/greatest-common-factor-online
https://www.mathmammoth.com/practice/gcf
https://www.mathgames.com/skill/6.52-greatest-common-factors-gcf
Love this one https://www.sheppardsoftware.com/math/fractions/greatest-common-factor-game/

LCM
https://www.mathgames.com/skill/6.53-least-common-multiples-lcm
https://mathkite.com/least-common-multiple-lcm/
https://mrnussbaum.com/least-common-multiple-online

From Nrich https://nrich.maths.org/factorsandmultiples

Recognizing Multiples
https://www.sheppardsoftware.com/math/multiples/catch-the-stars/

Printable in-person games
https://www.youtube.com/watch?v=yZ2qsDw_g7E this is a printable game, video how to play it

Another in person game https://occupymath.wordpress.com/2019/11/14/the-multiple-game/
The Product Game: https://connectedmath.msu.edu/covid-19-cmp-resources/resources-for-families/math-games/product-game/
How to play the product game video https://www.youtube.com/watch?v=6-FZ2Boug9s
A collection of rich tasks https://mrbartonmaths.com/topics/number-skills/factors-multiples-and-primes/richtasks.html

Multitrap: Printable paper pencil game
https://mathcurious.com/2021/01/27/multi-trap-multipication-facts-multiples-factors/
Multitrap google slides
https://docs.google.com/presentation/d/1iMcE5iOkjuyUdXvXy1nyxDZrJNvDm1z4YIZ0Sx49hjo/edit#slide=id.gb930cebdfa_0_385

Practice

Identifying multiples online worksheet
https://study.com/skill/practice/identifying-multiples-questions.html
https://www.topmarks.co.uk/Search.aspx?q=factors%20and%20multiples

https://www.multiplication.com/our-blog/jen-wieber/buzz-plus-number-game
https://calculate.org.au/2020/08/11/there-are-better-games-than-buzz/

https://helpingwithmath.com/worksheet/understanding-factors-and-multiples/

Circle all the multiples 8. Put a triangle around all the multiples of 3. What do you notice?

22	35	32	100	42
72	4	14	5	16
21	30	104	18	32
17	60	8	36	9
70	12	24	26	20
28	16	48	88	52
64	15	48	27	90
70	63	24	3	8

Teacher Version Circle all the multiples 8. Put a triangle around all the multiples of 3. What do you notice?

22	**35**	**32**	**100**	**42**
72	**4**	**14**	**5**	**16**
21	**30**	**104**	**18**	**32**
17	**60**	**8**	**36**	**9**
70	**12**	**24**	**26**	**20**
28	**16**	**48**	**88**	**52**
64	**15**	**48**	**27**	**90**
70	**63**	**24**	**3**	**8**

Students will notice that numbers with both circles and triangles are multiples of 24. Do 3 and 8 share any common factors? What is their least common multiple?

Circle all the multiples 6. Put a triangle around all the multiples of 4. What do you notice?

10	9	8	15	27
81	6	14	30	24
22	48	82	77	19
35	57	3	12	38
100	18	16	61	42
20	44	46	11	23
21	50	7	32	41
36	65	53	62	28

Teacher version Circle all the multiples 6. Put a triangle around all the multiples of 4. What do you notice?

10	9	8	15	27
81	6	14	30	24
22	48	82	77	19
35	57	3	12	38
100	18	16	61	42
20	44	46	11	23
21	50	7	32	41
36	65	53	62	28

Students will notice that some numbers have both a circle and rectangle. All numbers indicated are even. Why is that? Do students remember the rule for multiples of 6 (even numbers whose digits sum to three) ? Do 4 and 6 share any common factors? The LCM for 3 and 8 was their product, 24. What is the LCM for 4 and 6? Why is it not 24?

Circle all the multiples 8. Put a triangle around all the multiples of 3. What do you notice?

8 32 15 80

16

20

64 58

56

33 5

35 42

31

45

12

40

90 4 28 8

32

Circle all the multiples five. Put a triangle around all the multiples of 3. What do you notice?

27		15		6
	78	22	23	
24	20		81	
205		9	38	35
	100	44		15
		60	110	
51	150	30	52	39
	33	45		72

Circle all the multiples 4. Put a triangle around all the multiples of 3. What do you notice?

22	35	32	100	42
102	4	14	5	16
21	30	104	18	54
17	60	8	36	9
70	12	45	26	20
28	6	40	88	52
82	15	48	27	90
70	63	24	3	44

Teacher Version Circle all the multiples four. Put a triangle around all the multiples of 3. What do you notice?

22	35	32	100	42
102	4	14	5	16
21	30	104	18	54
17	60	8	36	9
70	12	45	26	20
28	6	40	88	52
82	15	48	27	90
70	63	24	3	44

Students should recognize that numbers that were both circled and triangle are multiples of both four and three and therefore 12. Have them list multiples of 12 if they don't pick this up
Did they recall that numbers divisible by three have digits that sum to a multiple of three?
Have them highlight all the multiples of two. Do they notice that all multiples of two are also multiples of four? Why is that? (use partial product to demonstrate)

MULTi-TRAP

132	55	7	66	12	30	84	72	10	8	60
64	12	80	49	3	110	14	22	9	24	28
96	44	21	35	63	15	70	32	40	56	77
1	33	18	50	77	2	48	90	6	80	27
72	60	42	20	25	44	32	48	21	64	81
45	4	30	5	144	63	108	66	50	35	24
14	40	54	24	36	11	56	45	8	22	36
20	88	27	96	70	49	18	28	54	99	5
9	16	42	77	15	84	55	33	25	16	56

The Product Game

1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
54	56	63	64	72	81

1 2 3 4 5 6 7 8 9

Object of the Game

To get four squares in a row—vertically, horizontally, or diagonally.

How to Play

1. To begin the game, Player 1 moves a marker (green square) to a number in the factor list of numbers 1-9 along the bottom of the game screen.
2. Player 2 then moves the other marker (green square) to any number in the factor list (including the number marked by Player 1). The product of the two marked numbers is determined, and that product is colored red for Player 2.
3. Player 1 moves *either* marker to another number, and the new product is colored blue for Player 1.
4. Players take turns moving a marker, and each product is marked red or blue, depending on which player made the product. However, if a product is already colored, the player does not get a square for that turn.
5. Play continues until one player wins, or until all squares have been colored.

Credit

This Product Game Investigation was adapted with permission and guidance from *Prime Time: Factors and Multiples*, Connected Mathematics Project, G. Lappan, J. Fey, W Fitzgerald, S. Friel and E. Phillips, Dale Seymour Publications (1996), pp. 17-25.

Exploration

Questions for Students

- Why did you pick <#> as your first product to play? Is there a better product you could have placed on the board?
- How many moves are your playing offensively vs. defensively? Can you make a move that is both?
- What factor(s) is(are) your favorite, and why?
- What factors give the most play options? Why do you think so?

Factors and Multiples Game

Instructions: You will need two different coloured markers.

Player 1 colours in any number less than 50.

Player 2 colours in a number which is a factor or multiple of the first number.

Player 1 now has to colour in a number that is a factor or multiple of the number player 2 coloured in.

Play continues until one player cannot go!

Remember: A "factor" of a number is any number that divides that number evenly without a remainder. *1,2,4,5,10, and 20 are* ***factors*** *of 20*

A multiple of a number is that number multiplied by any integer. They are the numbers you get when you skip count by a number, or recite it's multiplication table.
Multiples of 8 are 8, 16, 24, 32, 40, 48 etc

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

For flash cards: Print the following pages 1 – 4 double sided and cut out

Game suggestions: Keep score. If player correctly identifies GCF they add that quantity to their score. If they fail to identify a factor, or identify a factor that is not greatest, opponent gets the points.

Could also single side this and play as bingo

24 36	27 36	42 70	12 20
20 8	35 50	26 39	81 36
51 34	52 39	48 18	30 66
24 42	21 70	28 35	45 60

4	7	9	12
9	13	5	4
6	6	13	17
15	7	7	6

24 28	15 36	20 70	12 33
20 36	35 10	13 39	81 90
51 21	52 8	12 18	32 20
24 16	22 70	99 55	32 60

3	10	3	2
9	13	5	4
4	6	4	3
4	11	2	8